5. Stress-strain relations

The goal is to be able to relate stress and strain through a set of equations that involve the material properties. The result will be subject to the small strain / small rotation assumptions.

Some definitions:

Elastic material: stress state is a unique function of the strain state

Linearly elastic material: stress state is a linear function of the strain state

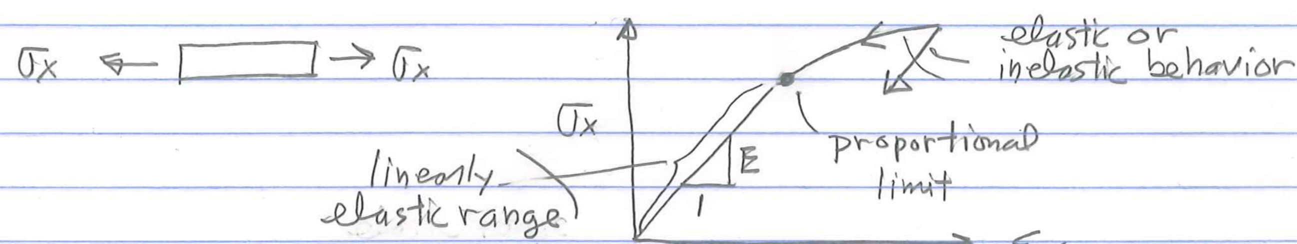
Inelastic material: Involves yielding, fracture, etc. The stresses depend on the history of the strains (loading history)

Homogeneous material: Properties are the same at every point in the body.

Isotropic material: Properties are independent of direction.

Consider first an isotropic material.

Take a small sample of material, apply σ_x and measure ε_x.



Thus, below the proportional limit,

$\varepsilon_x = \sigma_x / E$, where E is Young's modulus. You would also measure $\gamma_{xy} = \gamma_{yz} = \gamma_{xz} = 0$ and

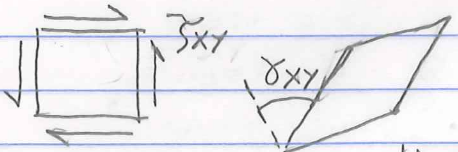
$$\varepsilon_y = \varepsilon_z = -\nu\, \varepsilon_x = -\nu\, \sigma_x / E$$ within this range,

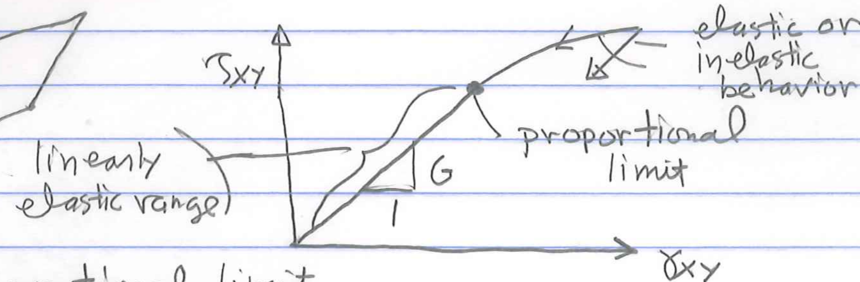
where ν is Poisson's ratio. ($\nu > 0$).

If σ_x, σ_y and σ_z are applied simultaneously, then superposition holds and

below the proportional limit
$$\left\{\begin{aligned}
\varepsilon_x &= \frac{\sigma_x}{E} - \frac{\nu}{E}\sigma_y - \frac{\nu}{E}\sigma_z \\
\varepsilon_y &= \frac{\sigma_y}{E} - \frac{\nu}{E}\sigma_x - \frac{\nu}{E}\sigma_z \\
\varepsilon_z &= \frac{\sigma_z}{E} - \frac{\nu}{E}\sigma_x - \frac{\nu}{E}\sigma_y
\end{aligned}\right.$$

Now take a small sample, apply τ_{xy} and measure γ_{xy}.





Thus, below the proportional limit, $\gamma_{xy} = \tau_{xy}/G$, where G is the shear modulus. Within this range, all the other strains are zero. Repeating the process with τ_{yz} and τ_{xz} leads to $\gamma_{yz} = \tau_{yz}/G$ and $\gamma_{xz} = \tau_{xz}/G$.

To summarize, for isotropic, linearly elastic material,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} \qquad (9)$$

The inverse relation exists and can be expressed as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} C_1 & C_2 & C_2 & 0 & 0 & 0 \\ C_2 & C_1 & C_2 & 0 & 0 & 0 \\ C_2 & C_2 & C_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \qquad C_1 = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \quad C_2 = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad (b)$$

It can also be shown that $G = \frac{E}{2(1+\nu)}$. This will be done later.

The above can also be used to derive a relation between pressure and volume strain.
Pressure is defined as $p = -\frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$, and volume strain is $e = \varepsilon_x + \varepsilon_y + \varepsilon_z$.
From (a)

$$e = -p/k \quad \text{where} \quad k = \frac{E}{3(1-2\nu)}$$

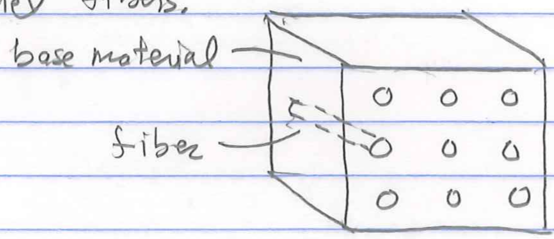
is the bulk modulus.
From the expression for k, the limit $\nu < \frac{1}{2}$ can be argued.

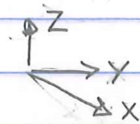
For a homogeneous material, E, ν (and G, k) are constant. For an inhomogeneous material, they are functions of x, y and z.

Consider now an anisotropic material. For the most general case, the 6×6 matrices in (a) and (b) are fully populated (but symmetric). For an orthotropic, linearly elastic material with directional aspects oriented with respect to x, y, z, equation (a) becomes symmetric

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{bmatrix} 1/E_x & -\nu_{yx}/E_y & -\nu_{zx}/E_z & 0 & 0 & 0 \\ -\nu_{xy}/E_x & 1/E_y & -\nu_{zy}/E_z & 0 & 0 & 0 \\ -\nu_{xz}/E_x & -\nu_{yz}/E_y & 1/E_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{xy} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{yz} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{xz} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} \quad (C)$$

An example of an orthotropic material would be the averaged properties of a fiber reinforced composite with parallel fibers.





Fibers parallel to x.

Usually, $E_{fiber} >> E_{base}$

If for a y-z cross-section the fiber area << than the base area, then

$$E_y, E_z \approx E_{base}$$
$$\nu_{xy}, \nu_{xz}, \nu_{yz}, \nu_{zy} \approx \nu_{base}$$
$$G_{xy}, G_{yz}, G_{xz} \approx G_{base}$$

and E_x and $\nu_{yx} = \nu_{zx}$ would depend on the fiber and base material properties and the amount of fibers. The above assumes the base material is isotropic.

Example:

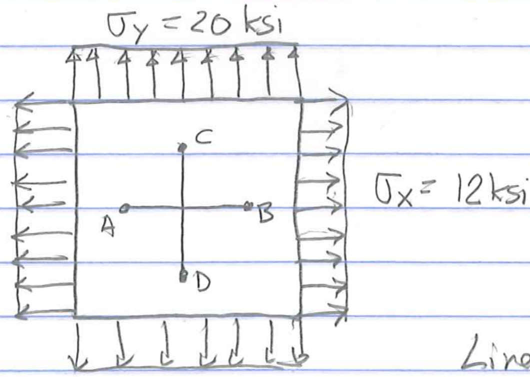


Plate is aluminum ($E = 10^4$ ksi) ($\nu = 0.25$)

Initial dimensions of plate are 15in × 15in × $\frac{3}{4}$ in thick

Lines A-B and C-D are initially 9 inches long.

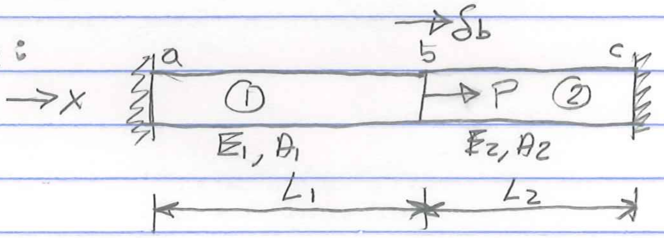
Find changes in lengths of AB and CD and the plate thickness and volume.

Note that the plate is under a state of uniform stress, with
$\sigma_x = 12\text{ ksi}$, $\sigma_y = 20\text{ ksi}$, $\sigma_z = \tau_{xy} = \tau_{yz} = \tau_{xz} = 0$ (plane stress)

$$\varepsilon_x = \sigma_x/E - \nu\sigma_y/E = \frac{1}{10^4\text{ ksi}}\left(12\text{ ksi} - \tfrac{1}{4}\,20\text{ ksi}\right) = 0.7\times 10^{-3}\text{ in/in}$$
$$\varepsilon_y = \sigma_y/E - \nu\sigma_x/E = \frac{1}{10^4\text{ ksi}}\left(20\text{ ksi} - \tfrac{1}{4}\,12\text{ ksi}\right) = 1.7\times 10^{-3}\text{ in/in}$$
$$\varepsilon_z = -\nu\sigma_x/E - \frac{\nu\sigma_y}{E} = \frac{1}{10^4\text{ ksi}}\left(-\tfrac{1}{4}\,12\text{ ksi} - \tfrac{1}{4}\,20\text{ ksi}\right) = -0.8\times 10^{-3}\text{ in/in}$$

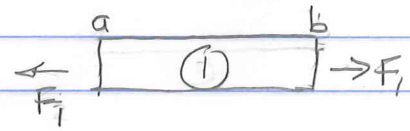
$$\Delta L_{AB} = 9\text{ in}\cdot\varepsilon_x = 6.3\times 10^{-3}\text{ in}$$
$$\Delta L_{CD} = 9\text{ in}\cdot\varepsilon_y = 15.3\times 10^{-3}\text{ in}$$
$$\Delta\text{ thickness} = 0.75\text{ in}\cdot\varepsilon_z = -0.6\times 10^{-3}\text{ in/in}$$
$$e = e_x + e_y + e_z = 1.6\times 10^{-3}\text{ in}^3/\text{in}^3$$
$$\Delta\text{ volume} = 15\text{ in}\cdot 15\text{ in}\cdot 0.75\text{ in}\cdot e = 0.270\text{ in}^3$$

Example:

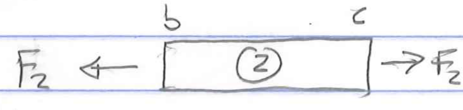


Linearly elastic material
Assume only σ_x exists and is uniform across the cross-section.

Find the deflection at b. (omit x subscript for clarity)
Denote internal forces in members (1) and (2) by F_1 and F_2 (tension positive).

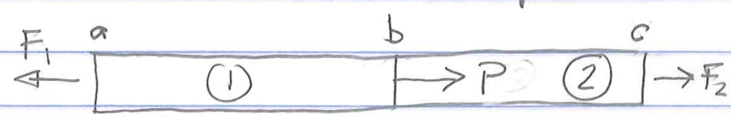


Member (1): $\varepsilon_1 = \delta_b / L_1$
$\sigma_1 = \varepsilon_1 E_1 = \delta_b E_1 / L_1$
$F_1 = \sigma_1 A_1 = \delta_b E_1 A_1 / L_1$



Member (2): $\varepsilon_2 = -\delta_b / L_2$
$\sigma_2 = \varepsilon_2 E_2 = -\delta_b E_2 / L_2$
$F_2 = \sigma_2 A_2 = -\delta_b E_2 A_2 / L_2$ F_2 will be negative (compression)

Consider the free body at right



$P + F_2 - F_1 = 0$
$P = F_1 - F_2 = \delta_b\left(E_1 A_1 / L_1 + E_2 A_2 / L_2\right)$
so $\delta_b = P / \left(E_1 A_1 / L_1 + E_2 A_2 / L_2\right)$

We can further find F_1 and F_2 by substituting for δ_b

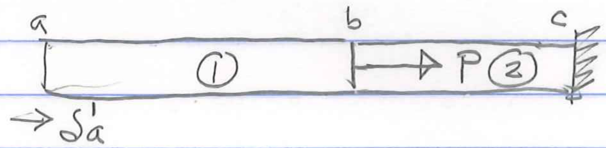
$$F_1 = P \frac{E_1 A_1 / L_1}{E_1 A_1/L_1 + E_2 A_2/L_2}$$

$$F_2 = -P \frac{E_2 A_2 / L_2}{E_1 A_1/L_1 + E_2 A_2/L_2}$$

Since the internal forces F_1 and F_2 depend on the E's and A's, this is a statically indeterminant problem.

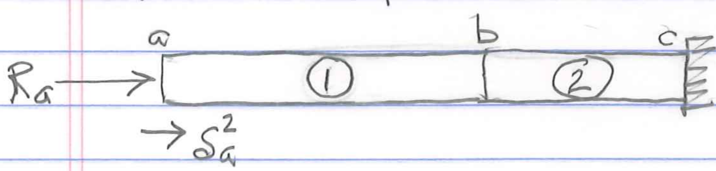
Superposition is an alternate way to solve this statically indeterminant problem. (Behavior must be linear for superposition to hold.)

Step 1: Remove left support and compute δ_a^1 from the existing loads.



$$\delta_a^1 = \frac{P}{A_2} \frac{1}{E_2} \cdot L_2 = \frac{P}{E_2 A_2 / L_2}$$

Step 2: Compute δ_a^2 for reaction force R_a at a



$$\delta_a^2 = R_a \cdot \left(\frac{1}{E_1 A_1/L_1} + \frac{1}{E_2 A_2/L_2} \right)$$

Step 3: Set the sum $\delta_a^1 + \delta_a^2 = 0$, and solve for R_a. The result is

$$R_a = -P \frac{E_1 A_1 / L_1}{E_1 A_1/L_1 + E_2 A_2/L_2} \quad \text{(so, acts to the left)}$$

Note that R_a equals $-F_1$ from above.